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The joy and mystery of natural numbers

At first glance, the natural numbers appear quite simple. Counting, 1,2,3,…, adding 3+2=5, and even multiplying, 3 x 4 = 12, are tasks mastered by children in early school age. At the same time the set of natural numbers hides some of the most celebrated problems in mathematics. We suggest two possible explanations for this dichotomy.

Addition vs. multiplication

Even if multiplication is normally considered as repeated addition, i.e 5+5+5+5+5+5=6x5, there is a structural gap between the two operations. Fermat claims in his famous last theorem, finally proved by Andrew Wiles in the nineties after 350 years of wondering, that a sum of two powers (of degree 3 or higher) of natural numbers can not itself be a power of a natural number of the same degree, so the addition disturbs the multiplicative structure, and we get lost.

Goldbachs conjecture, stated originally in 1742, says that any even number greater than 2, can be written as the sum of two prime numbers. Being a prime number is a purely multiplicative characteristic of a natural number. Adding two primes mixes the multiplicative and the additive structure in such a way that the outcome lacks tangible properties. Goldbachs claim is that any even number can be reached in this way, but nobody has been able to prove this statement.

Structure vs. randomness

The conflict between structure and randomness brings us to the second mystery of the natural numbers. If we pick a set of one or several natural numbers randomly, is it possible that this set satisfies some prescribed structural property? Szemerédis theorem is a good illustration of such a problem. The prescribed property in that case is the existence of arithmetic progressions (see separate article for a comprehensive description), and the question is how big the randomly picked set has to be to ensure the existence of arithmetic progressions of arbitrary length. Now, any structure you know about a natural number is given in terms of smaller numbers, i.e. on your way to 11, you will always pass 10, and the multiplicative characteristic of any number is given in terms of the prime number decomposition. On the other hand, we can randomly pick a collection of natural numbers, of infinite cardinality, and it is extremely difficult to trace back its structural properties.

Many elementary number theoretical problems can be classified as an addition/multiplication or a structure/random problem. We have mentioned Fermat`s last theorem, Goldbach`s conjecture and Szemerédis theorem. Other problems, like the twin prime mystery, can be viewed in this perspective. Twin primes are pairs of prime number of difference 2, such as 5 and 7, or 41 and 43. The question is whether there exists an infinite number of such pairs. Again, picking out prime numbers concerns the multiplicative structure of the natural numbers, whereas requiring the difference between them to be two has to do with the additive structure. We mix the two operations - , and get lost.