PETER D. LAX
ELEMENTS FROM HIS CONTRIBUTIONS TO MATHEMATICS

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1. INTRODUCTION

Peter D. Lax has given seminal contributions to several key areas of mathematics. His contributions are part of a long tradition where the interaction between mathematics and physics is at the core. Physics offers challenging problems that require intuition to solve. Mathematics can reveal deep inner strucures and properties, and rigorous proofs provide solid foundations for our knowledge. John von Neumann, who had considerable influence on Lax, concluded in 1945 that[1] "really efficient high-speed computing devices may, in the field of non-linear partial differential equations as well as in many other fields which are now difficult or entirely denied of access, provide us with those heuristic hints which are needed in all parts of mathematics for genuine progress." Lax stated in 1986 that[2] "[a]pplied and pure mathematics are more closely bound together today than any other time in the last 70 years". It is in this spirit that Lax has worked.

Peter D. Lax

In this short and nontechnical presentation we will focus on two areas, both within the theory of differential equations. Here we will address Lax's contributions where the applied aspects are dominant and have wide ranging consequences for our modern society. Thus, we will unfortunately not discuss his fundamental contributions to classical analysis and scattering theory, in particular, the development of the beautiful *Lax–Phillips scattering theory*.

The first topic is the theory of shock waves. Shock waves appear in many phenomena in everyday life. Most easily explained are shock waves coming from airplanes moving at supersonic speeds, or from explosions, but shocks also appear in phenomena involving much smaller velocities. Of particular interest is the flow of hydrocarbons in a porous medium, or to put it more concretely, the flow of oil in an oil

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[1] *Collected works of John v. Neumann*, vol. V, 1963, p. 1–32.

[2] Mathematics and its applications, *The Mathematical Intelligenzer* **8** (1986) 14–17.

reservoir. It is well-known that oil and water do not mix, and the interface between regions with oil and regions with water form what is mathematically defined as a shock. The dynamics of the shocks are vital in the exploitation of hydrocarbons from petroleum reservoirs. Even in everyday phenomena like traffic jams on heavily congested roads, we experience shock waves when there is an accumulation of cars. The shocks do not come from collisions of cars, but rather from a rapid change in the density of cars..

The second topic comes from the theory of solitons. The theory of solitons has a long and convoluted history, but now belongs at the heart of pure and applied mathematics with considerable consequences in several areas of technology. The theory originated in an obscure part of fluid dynamics. However, with, among other things, the discovery of the formulation of these problems using *Lax pairs*, new and startling connections between several different areas of mathematics were uncovered. Furthermore, soliton theory finds several applications in distinct areas of physics, for instance in quantum field theory and solid state physics, and in modeling biological systems. Finally, solitons are being applied to communication in optical fibers.

A more extensive discussion of several aspects of Peter Lax's contributions to mathematics can be found in [1]. An interview with him appeared in [2], and the full range of his contributions can be studied in his recently published selected works [3].

Before we return to a more detailed discussion of these topics, we will have to explain what a differential equation is.

2. What is a differential equation?

In order to discuss differential equations, we first have to introduce the derivative. Consider the situation when you are driving your car. On the odometer, you can measure the distance from your starting point, and knowing that, your position is determined. The distance you cover per unit of time is called the velocity, and that, of course, is what is displayed on the speedometer. Mathematically, the velocity is nothing but the derivative of the position. To put this in mathematical terms, we let x denote the position of the car, measured along the road from some starting point. It depends on time, t, so we write that $x = x(t)$. The velocity, which we denote by v and which depends on time, $v = v(t)$, is the change of position for a small time interval, and mathematically we call that the derivative[3] of x, and write $x'(t)$. Thus $v(t) = x'(t)$.

If a passenger in the car at each instant of time notes down the velocity, it should be possible to compute the position of the car at each point in

[3]To make it more precise, if you advance from position $x(t)$ at time t to position $x(t+s)$ during the time period s, the velocity at time t is approximately $(x(t+s)-x(t))/s$, and the approximation gets better the smaller time interval s you use. Mathematically, the velocity equals the limit of $(x(t+s)-x(t))/s$ as s tends to zero.

time if we know the time and place that the trip started. To put it more precisely, if we know the starting point x_0 (and synchronize our clocks so that we start at time $t = 0$), thus $x(0) = x_0$, and we know $v(t)$ for all t, we should be able to compute the position x as a function of time, that is, determine $x = x(t)$. To solve this problem we have to solve a differential equation, namely $x'(t) = v(t)$.

Differential equations are nothing but equations that involve derivatives. You may think that we are doing a lot out of a small problem. However, it turns out that all the fundamental laws of nature can be expressed as differential equations, as the following list displays

- Gravitation (Newton's law),
- Quantum mechanics (The Schrödinger equation),
- Electromagnetism (Maxwell's equations),
- Relativity (Einstein's equations),
- The motion of gases and fluids (The Navier–Stokes's equations).

The motion of planets, computers, electric light, the working of GPS (Global Positioning System), and the changing weather can all be described by differential equations.

Let us proceed to a more complicated example than the position and velocity of cars. Consider the heat in the room where you are sitting. At each point (x, y, z) in space and time t we let $T = T(x, y, z, t)$ denote the temperature. By assuming that heat flows from hot to cold areas proportional to the temperature difference, that heat does not disappear (which means that the room is completely isolated from the surroundings), and that there are no heat sources, one can derive that the temperature distribution is determined by the so-called heat equation, which reads

$$T_t = T_{xx} + T_{yy} + T_{zz}.$$

Here T_t means the derivative of the temperature with respect to the variable t, while T_{xx} denotes the derivative of the derivative, both with respect to the space variable x, and similarly for the remaining terms. Even simple problems give rise to difficult differential equations! Assuming that we know the initial temperature distribution, that is, we know $T = T(x, y, z, t)$ for $t = 0$, our intuition tells us that the temperature should be determined at all later times. This is called an initial-value problem. The mathematical challenge is to prove this assertion and describe a method to compute the actual temperature. In general terms this is the problem, but equations that are considerably more difficult than the heat equation form the core of Lax's contribution to differential equations.

Ideally, when you are given a differential equation, you want the problem to be well-posed in the sense that

- the problem should have at least one solution (existence of solution),
- the problem should have no more than one solution (uniqueness of solution),

- the solution should be stable with respect to perturbations (stability).

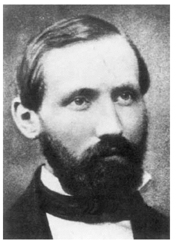
The first two conditions indicate that the problem should have a unique solution; the third condition states that a small change in the initial data should give a small change in the solution. Unfortunately, differential equations normally do not possess solutions that are given by formulas, and so we must add to our "wish list" that we should be able to find a way to compute the solution. The problems are often very complex and require high speed computers to determine an approximate, or numerical, solution. Solutions of differential equations may be very complicated, and there is no unified mathematical theory that covers all, or most, differential equations. Most of the interesting differential equations are nonlinear, where the sum of two solutions is not a solution, which further complicates matters. Different classes of differential equations require rather different methods, but even at this very general level, Lax has contributed two highly useful results that are described in all books in the area. The *Lax–Milgram theorem* provides a condition that stating that differential equations that can be described by an abstract variational problem, possess a unique solution. The *Lax equivalence principle* states that for a well-posed linear initial-value problem, any consistent numerical method is stable if and only if it is convergent. (The equivalence principle applies, for instance, to the heat equation.)

It is appropriate here to digress briefly on the interaction between mathematics and computers. Peter Lax has always been a strong proponent of the importance of computers to mathematics and vice versa, saying that[4] "[High speed computers'] impact on mathematics, both applied and pure, is comparable to the role of telescopes in astronomy and microscopes in biology". The logical construction of computers and their operating systems are mathematical by nature. But computers also serve as laboratories for mathematicians, where you can test your ideas. New mathematical relations can be discovered, and your hypotheses and assumptions can be disproved or made more likely by applying computers. Lax has given the example of the great mathematician G. D. Birkhoff who spent a lifetime trying to prove the ergodic hypothesis. If Birkhoff had had access to a computer and had tested the hypothesis on it, he would have seen that it cannot be true in general. On a more technical level, problems of modern technology like the simulation of systems as complex as airplanes, oil rigs, or the weather not only require very powerful computers, but also the development of new and better mathematical algorithms for their solution. It is a fact that, in broad terms, the development of

[4] The flowering of applied mathematics in America, *SIAM Review* **31** (1989) 533–541.

high speed computers (hardware) and the development of new numerical techniques (software) have contributed equally to the total performance we observe in simulations. Peter Lax himself has made penetrating contributions to the development of new mathematical methods that have enabled us to understand and simulate important phenomena.

3. Shock waves

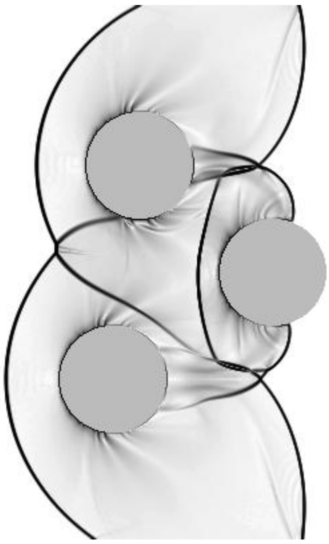


B. Riemann

In 1859 the brilliant German mathematician Berhard Riemann (1826–66) considered the following problem: If you have two gases at different pressures in a cylinder separated by a thin membrane, what happens if you remove the membrane? This problem has later been called the Riemann problem, and it turns out to be a very complicated question. The behavior of gases is well modeled by the Euler equations, which read[5]

$$\begin{aligned} \rho_t + (\rho v)_x &= 0, \\ (\rho v)_t + (\rho v^2 + P)_x &= 0, \\ E_t + (v(E+P))_x &= 0, \\ P &= P(\rho), \end{aligned}$$

where p, v, P, and E denote the density, velocity, pressure, and energy of the gas, respectively. This is a truely intricate system of equations that remains unsolved in the general case to this day.



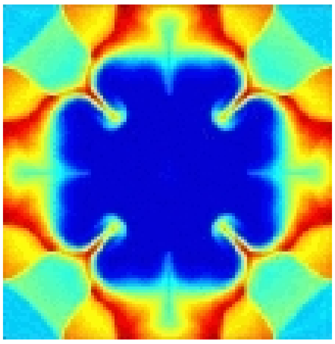
Flow of gas past three cylinders.

The Euler equations constitute a special case of a class of differential equations called hyperbolic conservation laws. The solutions of these equations are very complicated as the illustrations show. These equations are fundamental in several areas of applied science, for they express that a quantity is conserved. Examples abound because there is conservation of mass, momentum, and energy in isolated systems. In addition to the motion of gases, applications include the flow of oil in a petroleum reservoir. A less

[5]Riemann studied the simpler problem where the third equation, the one for the energy, is ignored. Subscripts indicate derivatives with respect to the variable given.

obvious example is the dynamics of cars on a highly congested road without exits or entries; here the conserved quantity is the number of cars.

The core of the problem with hyperbolic conservation laws, regardless of whether they describe traffic flow or the flow of oil in a petroleum reservoir, is that the solution develops singularities, or discontinuities, called shocks. Shocks correspond to very rapid transitions in density or pressure. Numerical methods have difficulty resolving these shocks, and the mathematical properties are very complicated. The mathematical models allow for more than one solution, and the selection principle, which is known as the entropy condition, for determining the one true physical solution is very complicated. Indeed, at this point Riemann erred and selected the wrong solution. The velocity of the shock was determined by the Scottish engineer, Rankine, and the French mathematician, Hugoniot, but it was left to Peter Lax in 1957 to come up with a simple criterion, now called the *Lax entropy condition*, that selects the true physical solution for general systems of hyperbolic conservation laws. The admissible shocks are called *Lax shocks*. The solution of the Riemann problem is now called the *Lax theorem*, and it is a cornerstone in the theory of hyperbolic conservation laws. His solution has stimulated extensive further research into different entropy conditions applicable to other systems. In particular, the fundamental existence result for the general initial-value problem posed by Glimm, uses the Lax theorem as a building block.

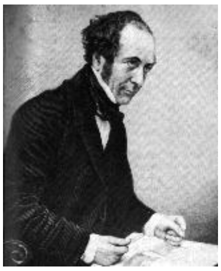


The pressure of a gas exploding in a box.

Once we have decided upon a selection principle, we still have to compute the solution. Here Peter Lax has introduced two of the standard numerical schemes for solving hyperbolic conservation laws, namely the so-called *Lax–Friedrichs scheme* and the *Lax–Wendroff scheme*. These schemes serve as benchmark tests for other numerical techniques and have served as a starting point for theoretical analysis. Indeed, the Lax–Friedrichs scheme was used by the Russian mathematician Oleĭnik in her constructive proof of the existence and uniqueness of solutions of the inviscid Burgers equation. Another highly useful result is the *Lax–Wendroff theorem*, which states the following: If a numerical scheme for a nonlinear hyperbolic conservation law converges to a limit, then we know that the limit at least is a solution of the equation. Together with Glimm, Lax proved deep results concerning the decay in time of solutions of systems of hyperbolic conservation laws.

Peter Lax's results in the theory of hyperbolic conservations laws are groundbreaking. They have resolved old problems, and have stimulated extensive new research in the field, and they are still at the core of the field.

4. SOLITONS

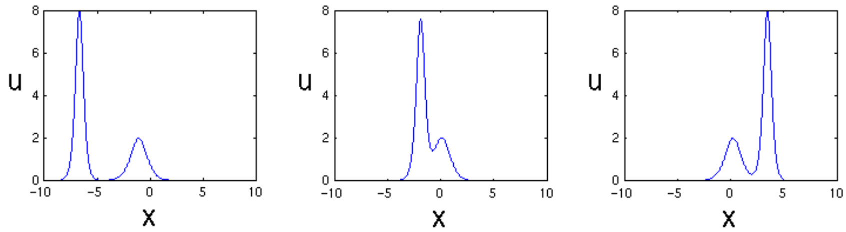


J. Scott Russell

The theory of solitons can be traced back to August 1834 when the Scottish engineer, John Scott Russell (1808–82), made the following observation: riding on his horse along a channel near Edinburgh, he observed a boat that was being pulled by horses along the channel. When the boat came to a halt, an isolated wave emanated from the bow, and Scott Russell was able to follow it for more than a kilometer. Contrary to what one would expect, the wave did not disperse, and its shape remained unchanged. Completely fascinated by the phenomena, which many people must have observed before him without noting its peculiarity, Scott Russell studied the waves that he called solitary waves for several years.

A modern re-enactment of the solitary wave.

His observations were controversial, and several eminent scientists, such as Airy and Stokes, were skeptical of his observations. However, with the derivation of a model for water waves by the Dutch mathematicians Korteweg and de Vries in 1895 that could indeed reproduce this behavior, solitary waves were established as a true, albeit rather specialized phenomena, of nature. The model they derived is now called the Korteweg–de Vries equation, KdV for short. To make a long story short, the KdV equation disappeared into oblivion for a long time, and only after renewed interest by Zabusky and Kruskal in 1965, was interest in the KdV equation revived. Through their analysis using numerical simulations, they discovered that the KdV had solutions that interacted like particles—they could collide and interact without changing shape. Zabusky and Kruskal coined these solutions "solitons" as they had particle-like properties like electrons, protons etc. (See the figure with two solitons.) It was now clear that the equation possessed deep structure and had potential for applications in several areas. In the landmark paper of 1967, Gardner, Greene, Kruskal, and Miura discovered an ingenious method, called the inverse scattering transform, for solving the KdV equation. Though the method was clearly a tour-de-force, it was highly attuned to to the peculiarities of the KdV equation. Several "miracles" made the method work. As part of their method, they studied an associated linear equation, for which several important quantities remained unchanged, or invariant, under the time evolution. Enter Peter Lax. He focused on the invariance properties of the linear problems and described a pair of operators, now called *Lax*



Two solitons, illustrated at three different times. The big soliton overtakes the small one. Their shapes are preserved.

pairs, that revealed the inner mechanism of the inverse scattering transform. When the Lax pair satisfies the *Lax relation*, it is indeed equivalent to the KdV equation. To make this connection more precise, let us first first write down the KdV equation, which reads[6]

$$u_t - 6uu_x + u_{xxx} = 0.$$

The Lax pair L, P is given by the operators[7]

$$L = -\partial_x^2 + u, \quad P = -4\partial_x^3 + 3u\partial_x + 3u_x,$$

with the property that the Lax relation[8]

$$L_t - (PL - LP) = u_t - 6uu_x + u_{xxx} = 0$$

holds. The Lax pair is so constructed that the differential operator on the left-hand side that is *a priori* a complicated differential operator, reduces to the KdV equation. Equations with properties like the KdV equation are called completely integrable.

With this deep and startling revelation, it was clear that the inverse scattering transform was not restricted to the KdV equation, and that Lax pairs for other differential equations of mathematical physics would now have to be looked at. Together with the zero-curvature formulation of Zakharov and Shabat, several of the important equations of mathematical physics suddenly turned out to be completely integrable, for instance, the sine-Gordon equation, the nonlinear Schrödinger equation, the massive

[6]The variable u corresponds to the distance from the surface of water to the bottom in Scott Russell's original observation.

[7]L and P are operators, that is, they are functions whose arguments in turn are functions. The operator ∂_x^n gives the nth derivative of a function with respect to the variable x.

[8]The derivation is as follows: The time invariance says that there is a unitary operator $U = U(t)$ such that $U^{-1}LU$ is time-independent, i.e., its derivative with respect to t vanishes. Postulating that U satisfies the first-order differential equation, $U_t = PU$ for some operator P, a short calculation shows that the Lax relation is satisfied.

Thirring system, the Boussinesq equation, the Kadomtsev–Petviashvili equation and the Toda lattice, to mention only a few.

The peculiar properties of these equations have had immense consequences in several areas of mathematics and physics as well as several areas of technology. One example can be mentioned here. There has been experimented using solitons for high-speed communication in optical fibers. The digital signal is coded using “ones" and “zeros", and we can let “ones" be represented by solitons. A key property of solitons is that they are exceptionally stable over very long distances. This offers the potential of considerably higher capacity in communication using optical fibers. Furthermore, the theory of solitons has revealed new and hitherto unknown relationships between various branches of mathematics.

Epilogue. Lax considers himself both a pure and an applied mathematician. His advice to young mathematicians is summarized in[9] “I heartily recommend that all young mathematicians try their skill in some branch of applied mathematics. It is a gold mine of deep problems whose solutions await conceptual as well as technical breakthroughs. It displays an enormous variety, to suit every style; it gives mathematicians a chance to be part of the larger scientific and technological enterprise. Good hunting!”

Acknowledgments. The portraits of Riemann and Scott Russell are from the MacTutor History of Mathematics Archive. The picture of the one soliton is from The Solitons Home Page. The simulations are provided by K.-A. Lie (SINTEF) and X. Raynaud (NTNU).

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