Yakov G. Sinai, Abel Prize Laureate 2014

THE ENTROPY OF A DYNAMICAL SYSTEM

Towards the end of the 1950s, the Russian mathematician Andrey Kolmogorov held a seminar series on dynamical systems at the Moscow University. A question often raised in the seminar concerned the possibility of deciding structural similarity between different dynamical systems. A young seminar participant, Yakov Sinai, presented an affirmative answer, introducing the concept of entropy of a dynamical system.

Let us start by going one decade back in time. In 1948 the American mathematician Claude E. Shannon published an article entitled ”*A Mathematical Theory of Communication*”. His idea was to use the formalism of mathematics to describe communication as a phenomenon. The purpose of all communication is to convey a message, but how this is done is the messenger's choice. Some will express themselves using numerous words or characters; others prefer to be more brief. The content of the information is the same, but the information density may vary. An example is the so-called SMS language. When sending an SMS message it is common to try to minimize the number of characters. The sentence ”I love You” consists of 10 characters, while ”I <3 U” consists of only 6, but the content of the two messages is the same.

Shannon introduced the notion of **entropy** to measure the density of information. To what extent does the next character in the message provide us with more information? High Shannon entropy means that each new character provides new information; low Shannon entropy indicates that the next character just confirms something we already know.

A dynamical system is a description of a physical system and its evolution over time. The system has many **states** and all states are represented in **the state space** of the system. A path in the state space describes the dynamics of the dynamical system.

A dynamical system may be **deterministic**. In a deterministic system no randomness is involved in the development of future states of the system. A swinging pendulum describes a deterministic system. Fixing the position and the speed, the laws of physics will determine the motion of the pendulum. When throwing a dice, we have the other extreme; **a stochastic** system. The future is completely uncertain; the last toss of the dice has no influence on the next.

In general, we can get a good overview of what happens in a dynamical system in the short term. However, when analyzed in the long term, dynamical systems are difficult to understand and predict. The problem of weather forecasting illustrates this phenomenon; the weather condition, described by air pressure, temperature, wind, humidity, etc. is a state of a dynamical system. A weather forecast for the next ten minutes is much more reliable than a weather forecast for the next ten days.

Yakov Sinai was the first to come up with a mathematical foundation for quantifying the complexity of a given dynamical system. Inspired by Shannon's entropy in information theory, and in the framework of Kolmogorovs Moscow seminar, Sinai introduced the concept of entropy for so-called measure-preserving dynamical systems, today known as **Kolmogorov-Sinai-entropy**. This entropy turned out to be a strong and far-reaching invariant of dynamical systems.

The Kolmogorov-Sinai-entropy provides a rich generalization of Shannon entropy. In information theory a message is an infinite sequence of symbols, corresponding to a state in the framework of dynamical systems. The shift operator, switching the sequence one step, gives the dynamics of the system. Entropy measures to what extent we are able to predict the next step in the sequence.

Another example concerns a container filled with gas. The state space of this physical system represents states of the gas, i.e. the position and the momentum of every single gas molecule, and the laws of nature determine the dynamics. Again, the degree of complexity and chaotic behaviour of the gas molecules will be the ingredients in the concept of entropy.

Summing up, the Kolmogorov-Sinai-entropy measures unpredictability of a dynamical system. The higher unpredictability, the higher entropy. This fits nicely with Shannon entropy, where unpredictability of the next character is equivalent to new information. It also fits with the concept of entropy in thermodynamics, where disorder increases the entropy, and the fact that disorder and unpredictability are closely related.

Kolmogorov-Sinai-entropy has strongly influenced our understanding of the complexity of dynamical systems. Even though the formal definition is not that complicated, the concept has shown its strength through the highly adequate answers to central problems in the classification of dynamical systems.

Mathematical formalism

Consider a dynamical system $(X, \mathcal{B}, \mu, T)$, and let $Q = \{Q_1, \ldots, Q_k\}$ be a partition of X into k measurable pair-wise disjoint pieces. We define the T-pullback of Q as

$$T^{-1}Q = \{T^{-1}Q_1, \ldots, T^{-1}Q_k\}$$

For two given partitions $Q = \{Q_1, \ldots, Q_k\}$ and $R = \{R_1, \ldots, R_m\}$, we define their refinement as

$$Q \vee R = \{Q_i \cap R_j \mid \mu(Q_i \cap R_j) > 0\}$$

Combining these two constructions we get the **refinement of an iterated pullback**;

$$\bigvee_{n=0}^{N} T^{-n}Q = \{Q_{i_0} \cap T^{-1}Q_{i_1} \cap \cdots \cap T^{-N}Q_{i_N}\}$$

The **entropy** of a partition Q is defined as

$$H(Q) = -\sum_{m=1}^{k} \mu(Q_m) \log \mu(Q_m)$$

and we put

$$h_\mu(T, Q) = \lim_{N \to \infty} \tfrac{1}{N} H(\bigvee_{n=0}^{N} T^{-n}Q)$$

The **Kolmogorov-Sinai-entropy** of $(X, \mathcal{B}, \mu, T)$ is defined as

$$h_\mu(T) = \sup_Q h_\mu(T, Q)$$

where the supremum is taken over all finite measurable partitions.